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Invariant quantization of the vector meson field

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Abstract. A method is given for computing the equal-time commutators of a Bose–Einstein field with constraints, directly from the lagrangian density, without necessitating the introduction of conjugate variables or solving equations of motion. The method is illustrated by the vector meson field.

1. General theory

If $\mathscr{L}(\phi^\alpha(x), \dot{\phi}^\alpha(x))$ is the lagrangian density and $L = \int \mathscr{L}\, d^3x$ is the Lagrange function of a classical deterministic Bose–Einstein field system, Hamilton's stationary action principle, $\delta \int L\, dt = 0$, leads to the equations of motion and constraint

$$\frac{\eth \mathscr{L}}{\eth \phi^\alpha(x)} - \frac{d}{dt}\frac{\eth \mathscr{L}}{\eth \dot{\phi}^\alpha(x)} = 0, \tag{1}$$

where $\eth$ is the variational derivative, and d is the total derivative at fixed x. If we consider variations in which the end points of the orbit vary we obtain the result

$$\delta \int_{t_1}^{t_2} L\, dt = \delta \int_{t_1}^{t_2} \int \mathscr{L}\, d^3x\, dt = \left[\int \{\pi_\alpha(x)\, \delta\phi^\alpha(x) - \mathscr{H}(x)\, \delta t\}\, d^3x \right]_{t_1}^{t_2} \tag{2}$$

where

$$\pi_\alpha(x) = \frac{\eth \mathscr{L}(\phi^\alpha(x), \dot{\phi}^\alpha(x))}{\eth \dot{\phi}^\alpha(x)}, \qquad \mathscr{H}(x) = \pi_\alpha(x)\dot{\phi}^\alpha(x) - \mathscr{L}(x). \tag{3}$$

$\mathscr{H}$ is the hamiltonian density (Ziman 1969) and the hamiltonian of the system is given by $H = \int \mathscr{H}\, d^3x$. We can define the Lagrange bracket, $\delta_1 \times \delta_2$, by

$$\delta_2 \times \delta_1 = \int \{\delta_1 \pi_\alpha(x'') \,.\, \delta_2 \phi^\alpha(x'') - \delta_2 \pi_\alpha(x'') \,.\, \delta_1 \phi^\alpha(x'') - \delta_1 \mathscr{H}(x'') \,.\, \delta_2 t + \delta_2 \mathscr{H}(x'') \,.\, \delta_1 t\}\, d^3x'' = -\delta_1 \times \delta_2. \tag{4}$$

It will be sufficient to consider only variations for which $\delta_1 t = \delta_2 t = 0$. It can be shown (Allcock 1973) that any functional $G_1[\phi^\alpha(x, t), \dot{\phi}^\alpha(x, t)]$ of the orbit variables $\phi^\alpha(x, t), \dot{\phi}^\alpha(x, t)$ at a fixed time t generates a Lagrange-bracket-preserving transformation δ_1 of the orbit variables at the time t, according to the equation

$$\delta G_1 = \delta_1 \times \delta, \tag{5}$$

where δ is an arbitrary displacement of the orbit variables at the time t and both δ

and δ_1 obey the constraints of the system. $\delta_1 \times \delta$ is defined in a similar manner to $\delta_1 \times \delta_2$. δ_1 so defined is called an infinitesimal contact transformation of the orbit variables at time t.

It can then be shown (Allcock 1973) that if G_1 and G_2 are any two functionals of the field variables and δ_1 and δ_2 are the contact transformations they respectively induce, then the equal-time Poisson bracket of G_1 and G_2 is given by

$$\left(G_1, G_2\right) = \delta_1 \times \delta_2 = \delta_2 G_1 = -\delta_1 G_2 \tag{6}$$

and hence, if G_1 and G_2 are functionals of the variables of the corresponding quantum system, their equal-time commutator (Dirac 1947) is given by

$$[G_1, G_2] = \mathrm{i}\delta_1 \times \delta_2, \tag{7}$$

to within possible ambiguities arising from the inequivalence of different quantal factor orderings.

To allow for the fact that $\delta\phi^\alpha$ and $\delta\dot{\phi}^\alpha$ obey the equations of constraint, $\psi_m(x) = 0$, of the system, we introduce continuous Lagrange multipliers $\lambda^m(x)$ and write equation (5) as

$$\delta G_1 + \int \lambda^m(x')\,\delta\psi_m(x')\,\mathrm{d}^3x' = \delta_1 \times \delta. \tag{8}$$

We can then regard δ as being completely arbitrary.

We now define functional derivatives $dG_1/d\phi^\alpha(x)$ and $dG_1/d\dot{\phi}^\alpha(x)$ of G_1 by

$$\delta G_1 = \int \left(\frac{dG_1}{d\phi^\alpha(x')} . \delta\phi^\alpha(x') + \frac{dG_1}{d\dot{\phi}^\alpha(x')} . \delta\dot{\phi}^\alpha(x')\right) \mathrm{d}^3x'. \tag{9}$$

Similarly we define functional derivatives of $\psi_m(x')$ and $\pi_\alpha(x'')$. Substituting into (7) and equating coefficients of $\delta\phi^\alpha(x'')$ and $\delta\dot{\phi}^\alpha(x'')$ we obtain the two equations

$$\frac{dG_1}{d\phi^\alpha(x'')} + \int \lambda^m(x') . \frac{d\psi_m(x')}{d\phi^\alpha(x'')}\,\mathrm{d}^3x'$$
$$= \int \left[\frac{d\pi_\beta(x')}{d\phi^\alpha(x'')} . \delta_1\phi^\beta(x') - \left(\frac{d\pi_\alpha(x'')}{d\phi^\beta(x')} . \delta_1\phi^\beta(x') + \frac{d\pi_\alpha(x'')}{d\dot{\phi}^\beta(x')} . \delta_1\dot{\phi}^\beta(x')\right)\right] \mathrm{d}^3x' \tag{10}$$

$$\frac{dG_1}{d\dot{\phi}^\alpha(x'')} + \int \lambda^m(x') . \frac{d\psi_m(x')}{d\dot{\phi}^\alpha(x'')}\mathrm{d}^3x' = \int \frac{d\pi_\beta(x')}{d\dot{\phi}^\alpha(x'')} . \delta_1\phi^\beta(x')\,\mathrm{d}^3x'. \tag{11}$$

The equations of constraint become $\delta_1\psi_m = 0$, or explicitly

$$\int \left(\frac{d\psi_m}{d\phi^\alpha(x')} . \delta_1\phi^\alpha(x') + \frac{d\psi_m}{d\dot{\phi}^\alpha(x')} . \delta_1\dot{\phi}^\alpha(x')\right) \mathrm{d}^3x' = 0. \tag{12}$$

A similar set of equations hold for $\delta_2\phi^\alpha$ and $\delta_2\dot{\phi}^\alpha$ generated by the functional G_2. It can be shown (Allcock 1974) that the rank of the system of equations (9)–(11) is maximal for a deterministic system, and therefore the solution is unique. Hence, we can compute the equal-time Poisson brackets for any two functionals of the field variables by (6).

2. Real vector meson (Proca–Wentzel–Kemmer) field

Notation. Unprimed variables are functions of x, primed variables of x', etc, $\partial_\mu \equiv \partial/\partial x^\mu$, $\partial'_\mu \equiv \partial/\partial x'^\mu$ etc. $g_{\mu\nu}$ is the metric tensor with diagonal elements $1, -1, -1, -1$. Roman indices run from 1 to 3, Greek indices from 0 to 3. ($\hbar = c = 1$.)

We take the lagrangian density of the vector meson field (Wentzel 1949, Proca 1936, Pauli 1941, Kemmer 1938, Schwinger 1970) to be

$$\mathscr{L} = -\tfrac{1}{4}(\partial_\mu\phi_\nu - \partial_\nu\phi_\mu)(\partial^\mu\phi^\nu - \partial^\nu\phi^\mu) + \tfrac{1}{2}m^2\phi_\mu\phi^\mu.$$

The stationary action principle leads to the equations of motion

$$(-\partial^\mu\partial_\mu - m^2)\phi^\nu = 0 \tag{13}$$

and the primary equation of constraint

$$\psi_1 \equiv \partial_j\dot{\phi}^j - (m^2 + \partial_j\partial^j)\phi^0 = 0. \tag{14}$$

By taking the time derivative of ψ_1 (Allcock 1974) we have the secondary equation of constraint

$$\psi_2 \equiv \partial_\nu\phi^\nu = 0. \tag{15}$$

Differentiation of ψ_2 with respect to time yields no further information about ϕ^ν and $\dot{\phi}^\nu$, and so ψ_1 and ψ_2 are the only two constraints on the system.

From (3) we define $\pi_\nu(x)$ by

$$\begin{aligned} \pi_j(x) &= \partial_j\phi^0(x) + \dot{\phi}^j(x) \\ \pi_0(x) &\equiv 0. \end{aligned} \tag{16}$$

The second of equations (16) leads to well known difficulties (Wentzel 1949) in the usual formalism which can only be dealt with by *ad hoc* manipulations. The present method deals with these difficulties without recourse to heuristic methods.

By comparison with (8) we have

$$\begin{aligned} &\frac{d\pi''_j}{d\phi^{\nu'}} = -\partial'_j\delta^3(x'-x'')\delta^0_\nu, && \frac{d\pi''_j}{d\dot{\phi}^{\nu'}} = \delta^3(x'-x'')\delta^j_\nu, \\ &\frac{d\psi'_1}{d\phi^{\nu''}} = -(m^2 + \partial''_j\partial^{j''})\delta^3(x'-x'')\delta^\nu_0, && \frac{d\psi'_1}{d\dot{\phi}^{\nu''}} = -\partial''_\nu\delta^3(x'-x'')(1-\delta^{(\nu)}_0), \\ &\frac{d\psi'_2}{d\phi^{\nu''}} = -\partial''_\nu\delta^3(x'-x'')(1-\delta^{(\nu)}_0), && \frac{d\psi'_2}{d\dot{\phi}^{\nu''}} = \delta^3(x'-x'')\delta^0_\nu. \end{aligned} \tag{17}$$

Let $G_1 \equiv \phi^\sigma(x)$, we have

$$\frac{dG_1}{d\phi^{\nu''}} = \delta^\sigma_\nu\delta^3(x-x''), \qquad \frac{dG_1}{d\dot{\phi}^{\nu''}} = 0. \tag{18}$$

Substituting the above values into equations (10)–(12) we find that the displacements

$\delta_1\phi^\nu$ and $\delta_1\dot{\phi}^\nu$ generated by $G_1 \equiv \phi^\sigma(x)$ are specified by the four equations

$$\begin{aligned}
&\delta^\sigma_\nu\delta^3(x-x'')-(m^2+\partial''_j\partial^{j''})\lambda''_1\delta^0_\nu-\partial''_\nu\lambda''_2(1-\delta^{(\nu)}_0) \\
&\qquad = -\partial''_j\delta_1\phi^{j''}\delta^0_\nu-\partial''_\nu\delta_1\phi^{0''}(1-\delta^{(\nu)}_0)-\delta_1\dot{\phi}^{\nu''}(1-\delta^{(\nu)}_0), \\
&-\partial''_\nu\lambda''_1(1-\delta^{(\nu)}_0)+\lambda''_2\delta^0_\nu = \delta^\nu_j\delta_1\phi^{j''}, \\
&\partial''_j\delta_1\phi^{j''}+\delta_1\dot{\phi}^{0''} = 0, \\
&-(m^2+\partial''_j\partial^{j''})\delta_1\phi^{0''}+\partial''_j\delta_1\dot{\phi}^{j''} = 0,
\end{aligned} \tag{19}$$

which we solve to obtain

$$\lambda''_1 = \frac{1}{m^2}\delta^\sigma_0\delta^3(x-x''), \qquad \lambda''_2 = 0,$$

$$\delta_1\phi^{k''} = -\frac{1}{m^2}\partial''_k\delta^3(x-x'')\delta^\sigma_0, \qquad \delta_1\phi^{0''} = -\frac{1}{m^2}\delta^\sigma_k\partial''_k\delta^3(x-x''), \tag{20}$$

$$\delta_1\dot{\phi}^{k''} = -\delta^\sigma_j\left(\delta^j_k+\frac{1}{m^2}\partial''_k\partial^{j''}\right)\delta^3(x-x''), \qquad \delta_1\dot{\phi}^{0''} = \frac{1}{m^2}\partial''_k\partial''_k\delta^3(x-x'')\delta^\sigma_0. \tag{21}$$

If we now take a second generator $G_2 \equiv \dot{\phi}^\tau(x')$, and again substitute into equations (10)–(12), using θ_1 and θ_2 as Lagrange multipliers, we obtain a set of equations similar to (18), which yield the results

$$\theta''_1 = \frac{1}{m^2}\delta^\tau_k\partial''_k\delta^3(x'-x''), \qquad \theta''_2 = -\delta^\tau_0\,\delta^3(x'-x''),$$

$$\delta_2\phi^{k''} = \delta^\tau_i\left(\delta^i_k+\frac{1}{m^2}\partial''_k\partial^{i''}\right)\delta^3(x'-x''), \qquad \delta_2\phi^{0''} = \frac{1}{m^2}\delta^\tau_0\partial''_k\partial^{k''}\delta^3(x'-x''), \tag{22}$$

$$\begin{aligned}
\delta_2\dot{\phi}^{k''} &= -\delta^\tau_0\partial''_k\left(1+\frac{1}{m^2}\partial''_j\partial^{j''}\right)\delta^3(x'-x''), \\
\delta_2\dot{\phi}^{0''} &= -\delta^\tau_i\partial''_k\left(\delta^i_k+\frac{1}{m^2}\partial''_k\partial^{i''}\right)\delta^3(x'-x'').
\end{aligned} \tag{23}$$

Let $G_3 \equiv \phi^\tau(x')$. From (6) we have

$$\begin{aligned}
\left(G_1, G_3\right) &\equiv \left(\phi^\sigma(x), \phi^\tau(x')\right) = \delta_1\times\delta_3 = -\delta_1 G_3 = -\delta_1\phi^\tau(x') \\
&= -\frac{1}{m^2}(\delta^\tau_0\delta^\sigma_k+\delta^\tau_k\delta^\sigma_0)\partial_k\delta^3(x-x') \qquad \text{(from (20))} \\
&= -\mathrm{i}[\phi^\sigma(x), \phi^\tau(x')].
\end{aligned} \tag{24}$$

Similarly, with $G_1 \equiv \phi^\sigma(x)$ and $G_2 \equiv \dot{\phi}^\tau(x')$ we have

$$\begin{aligned}
\left(G_1, G_2\right) &\equiv \left(\phi^\sigma(x), \dot{\phi}^\tau(x')\right) = \delta_1\times\delta_2 = -\delta_1 G_2 = -\delta_1\dot{\phi}^\tau(x') \\
&= -\frac{1}{m^2}\delta^\sigma_0\delta^\tau_0\nabla^2\delta^3(x-x')-\delta^\sigma_k\delta^\tau_j\left(g_{kj}+\frac{1}{m^2}\partial_k\partial_j\right)\delta^3(x-x') \qquad \text{(from (21))} \\
&= -\mathrm{i}[\phi^\sigma(x), \dot{\phi}^\tau(x')].
\end{aligned} \tag{25}$$

With $G_2 \equiv \dot{\phi}^\tau(x')$ and $G_4 \equiv \dot{\phi}^\sigma(x)$ we have

$$\left(G_4, G_2\right) \equiv \left(\dot{\phi}^\sigma(x), \dot{\phi}^\tau(x')\right) = \delta_2 \times \delta_4 = -\delta_2 G_4 = -\delta_2 \dot{\phi}^\sigma(x')$$

$$= -(\delta_0^\tau \delta_j^\sigma + \delta_0^\sigma \delta_j^\tau)\partial_j \left(1 - \frac{\nabla^2}{m^2}\right)\delta^3(x-x') \qquad \text{(from (23))}$$

$$= -\mathrm{i}[\dot{\phi}^\sigma(x), \dot{\phi}^\tau(x')]. \tag{26}$$

The commutation relations given in (24)–(26) are equivalent to those given in the standard texts on the vector meson field (eg Wentzel 1949, Kemmer 1938). The method given is no more complicated than one would expect, in view of the results, and gives the quantum relations directly without the *ad hoc* methods of the canonical formalism. An application of the method will be given for the Rarita–Schwinger field for a particle with spin $\frac{3}{2}$ in a subsequent paper.

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